Recombinant Growth

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Theoretical Contribution

Introduction

When someone invents something new, what exactly is happening? With a few exceptions, economists typically evade this question and assume research and design (R&D) effort somehow gets translated into new ideas (at least, with some probability). Weitzman attempts to push beyond this evasion and demonstrate that how we understand the innovation process can have important macroeconomic consequences.

Weitzman argues that innovation can be understood as a *combinatorial process*. Innovation is a process where two pre-existing ideas or technologies are combined and, if you pour in sufficient R&D resources and get lucky, a new idea or technology is the result. Weitzman's own example is Edison's hunt for a suitable material to serve as the filament in the light bulb. Edison combined thousands of different materials with the rest of his lightbulb apparatus before hitting upon a combination that worked. But the lightbulb isn't special: essentially any idea or technology you can think of can also be understood as a novel configuration of pre-existing parts.

An important point is that once you successfully combine two components, the resulting new idea becomes a component you can combine with others. To stretch the lightbulb example, once the lightbulb had been invented, new inventions that use lightbulbs as a technology could be invented: things like desk lamps, spotlights, headlights, and so on.

For example, suppose that you have three ideas *a*, *b*, and *c*. You can combine these in three different ways:

1. *a-b*
2. *a-c*
3. *b-c*.

For simplicity, lets assume all three combinations generate new ideas. Now you have six ideas: *a*, *b*, *c*, *ab*, *ac*, and *bc.* These can be combined in 15 different ways:

1. *a-b*
2. *a-c*
3. *a-ab*
4. *a-ac*
5. *a-bc*
6. *b-c*
7. *b-ab*
8. *b-ac*

9. *b-bc*
10. *c-ab*
11. *c-ac*
12. *c-bc*
13. *ab-ac*
14. *ab-bc*
15. *ac-bc*

Of these, you will already have tried *a-b*, *a-c*, and *b-c*, but the other 12 are new. If all 12 generate new ideas, then you now have 18 ideas total, which can be combined in 153 different ways! Clearly, combinatorial processes can grow quite quickly.

Result #1: Combinatorial Processes Eventually Overtake Exponential Ones

How quickly can a combinatorial process grow? It is not uncommon to use "exponential growth" to refer to the notion that growth rapidly accelerates. But Weitzman's first result is to show that given a large enough starting set of ideas, combinatorial processes like the one described above *eventually* grow faster than *any* exponential process.

Weitzman's proof is beyond the scope of this research digest, but a numerical example will illustrate the point. Define two processes to describe the growth rate of ideas, each of which starts with 100 ideas.

Period	Combinatorial Process	Exponential Process
0	100	100

For the combinatorial process, we will assume only 1% of combinations successfully generate new ideas (for comparison, we assumed 100% of combinations generate new ideas in our previous example). For the exponential process, we will assume it doubles in size every period.

In the first, period, the exponential process doubles in size from 100 to 200. For the combinatorial process, each idea can be paired with 99 other ideas, for a total of 100 x 99 = 9,900 possible pairs. However, this is actually twice the correct number, because it double counts ideas (for example, combining idea #1 with idea #99, and idea #99 with idea #1). In general, if you have N ideas, then there are $N(N-1)/2$ unique "pairs" of ideas. In our example, this implies 4,950 possible ways to combine the original 100 ideas. Of these, only 1% generate new ideas, so (rounding down), 49 new ideas are added to the pile.

Period	Combinatorial Process	Exponential Process
0	100	100
1	149	200

In the next period, the exponential process doubles in size to 400. For the combinatorial process, the 149 ideas have 11,026 possible combinations. However, of these, we already tried 4,950 in the last period, so there is "only" 6,076 new combinations possible. Since only 1% of them generate new ideas, (rounding down) there are 60 new ideas added to the combinatorial process.

Period	Combinatorial Process	Exponential Process
0	100	100
1	149	200
2	209	400

At this point, we can see that the exponential process is now nearly twice as large as the combinatorial process. We continue the process for another few stages:

Period	Combinatorial Process	Exponential Process
0	100	100
1	149	200
2	209	400
3	316	800
4	596	1,600
5	1,871	3,200

In the subsequent three periods, the exponential process initially appeared to pull away from the combinatorial one. In period 3, it was 2.5 times as large as the combinatorial process, and in period 4, it was 2.7 times as large. However, by period 5, its lead had shrunk a bit, and it was only 1.7 times as large as the combinatorial process.

In the next period, the exponential process doubles to 6,400. In the combinatorial process, the 1,871 ideas can be combined in a staggering 1.75 million different ways, and nearly all of these pairs of ideas are new. Although only 1% of the combinations result in new ideas, this still results in 15,720 new ideas:

Period	Combinatorial Process	Exponential Process
0	100	100
1	149	200
2	209	400
3	316	800
4	596	1,600
5	1,871	3,200
6	17,591	6,400

By period 6, the combinatorial process has left the exponential process in the dust. From this point forward, its lead widens at an extremely rapid pace.

Period	Combinatorial Process	Exponential Process
0	100	100
1	149	200

2	209	400
3	316	800
4	596	1,600
5	1,871	3,200
6	17,591	6,400
7	1,547,225	12,800
8	11,969,518,363	25,600

Weitzman mathematically proves that this kind of result will always hold, no matter how quickly the exponential process grows and no matter how rare it is for a combination of ideas to generate a new one.

An important caveat is that you have to have enough ideas to start with. To see what this means, suppose that instead of 100 ideas, we only started with 25:

Period	Combinatorial Process	Exponential Process
0	25	25

In the first period, the exponential process doubles in size to 50. There are 25 x 24 /2 = 300 possible combinations in the combinatorial process. Since only 1% of them generate new ideas, we obtain 3 new ideas.

Period	Combinatorial Process	Exponential Process
0	25	25
1	28	50

In the next period, the exponential process doubles in size to 100. There are 28 x 27 / 2 = 378 possible combinations in the combinatorial process, but we tried 300 of them in the previous period. This means there are only 78 new combinations. Since only 1% of them work, if we round down, we don't add any new ideas.

Period	Combinatorial Process	Exponential Process
0	25	25
1	28	50
2	28	100

At this point, the combinatorial process is dead. All possible combinations have been tried, and so the process will remain stuck at 28 ideas forever. Meanwhile, the exponential process will grow forever.

To summarize, if you have enough initial ideas to get the combinatorial process on its growth trajectory, it will eventually overtake *any* exponential growth process. If there are not enough initial ideas though, the combinatorial process will eventually peter out and stop growing at all though.

Result #2: Economic growth is still exponential in the long run

This result would seem to imply that economic growth should get faster and faster. Weitzman's second result is to show this is actually **not** the case. Instead, in the long-run economic growth converges to a steady exponential rate. This is because Weitzman assumes innovation requires R&D effort, in addition to the raw material of new ideas. In the long run, growth becomes constrained by the resources we have available to devote to R&D, not by the supply of possible ideas. Again, the proof is beyond the scope of this digest, but the basic idea is easy to illustrate numerically.

Consider the following extremely simplified model. Innovation requires three elements: two ideas to combine, plus one unit of output to cover R&D expenses. When all three of these elements are in place, there is a 1% probability that innovation will result in a new idea.

We abstract away from labor and capital, and assume the only productive input to the economy is ideas. These ideas translate into GDP in an extremely simple manner: GDP is just equal to one hundred times the number of ideas. This is an economy that is completely focused on growing as rapidly as possible, and so it will devote as many resources as it can to R&D.

In this economy, the growth rate is GDP in the current period, divided by GDP in the previous period. Assume once again that we start with 100 ideas, which means GDP starts at 10,000.

Period	GDP	# of ideas	# of possible combinations	# of investigated combinations	Share of GDP devoted to R&D	Growth rate
0	10,000	100	4,950	4,950	50%	-

As we noted in the previous example, starting with 100 ideas, there are 4,950 possible ideas to investigate. R&D over all these possible ideas costs 4,950 units of GDP, so that nearly 50% of the economy is devoted to R&D. As noted in the previous example, since only 1% of the investigated ideas will turn out to generate new ideas, only 49 new ideas are discovered.

Period	GDP	# of ideas	# of possible combinations	# of investigated combinations	Share of GDP devoted to R&D	Growth rate
0	10,000	100	4,950	4,950	50%	-
1	14,900	149	6,076	6,076	41%	49%

GDP has risen to 14,900, implying a growth rate of 49%. As noted in the previous section, given 149 ideas, there are 6,076 new combinations of ideas to investigate. R&D for this costs 6,076 units of GDP, so that 41% of the economy is devoted to R&D. As noted previously, this investigation yields 60 new ideas. Below, we iterate the economy forward a few steps.

Period	GDP	# of ideas	# of possible combinations	# of investigated combinations	Share of GDP devoted to R&D	Growth rate
0	10,000	100	4,950	4,950	50%	-
1	14,900	149	6,076	6,076	41%	49%
2	20,900	209	10,710	10,710	51%	40%
3	31,600	316	28,034	28,034	89%	51%

4	59,600	596	127,540	59,600	100%	88%

Up through period 3, things are much as they have been previously. But in period 4, a new constraint hits. In period 4, there are 596 ideas. These can be paired up to yield 177,310 combinations. Of these 49,770 have been investigated in previous periods, leaving 127,540 new pairs of ideas to investigate.

The R&D costs to investigate all 127,540 new ideas exceeds the GDP of the economy. At most, R&D resources equal to 59,600 can be expended on R&D, meaning the economy can only investigate 59,600 pairs. Since 1% of these yield new ideas, the number of ideas increases by another 596 to 1,192.

Period	GDP	# of ideas	# of possible combinations	# of investigated combinations	Share of GDP devoted to R&D	Growth rate
0	10,000	100	4,950	4,950	50%	-
1	14,900	149	6,076	6,076	41%	49%
2	20,900	209	10,710	10,710	51%	40%
3	31,600	316	28,034	28,034	89%	51%
4	59,600	596	127,540	59,600	100%	88%
5	119,200	1,192	600,466	119,200	100%	100%

With 1,192 ideas, there are over 700,000 potential combinations, and only 109,370 have been investigated. Once again, the cost of R&D for all potential idea combinations exceeds GDP. If 100% of the economy is devoted to R&D, 119,200 idea-combinations can be investigated, yields 1,192 new ideas. Again the economy doubles.

Period	GDP	# of ideas	# of possible combinations	# of investigated combinations	Share of GDP devoted to R&D	Growth rate
0	10,000	100	4,950	4,950	50%	-
1	14,900	149	6,076	6,076	41%	49%
2	20,900	209	10,710	10,710	51%	40%
3	31,600	316	28,034	28,034	89%	51%
4	59,600	596	127,540	59,600	100%	88%
5	119,200	1,192	600,466	119,200	100%	100%
6	238,400	2,384	2,611,966	238,400	100%	100%

This constraint will continue to bind for as long into the future as we wish to go. In the long run, the binding constraint to growth becomes the resources available to "process" ideas, not the number of possible ideas themselves. In this example, 100% of GDP was devoted to R&D, which led to an annual growth rate of 100%. More generally though, Weitzman shows the long-run growth rate will be proportional to the share of the economy devoted to R&D. Exponential growth takes over in the long run, even though innovation is combinatorial.

Additional Results

Much of Weitzman's paper demonstrates that the above results continue to hold as the assumptions are modified. He shows that embedding a combinatorial innovation process into a more realistic model of the economy, including labor and capital, does not overturn result #2. Neither is it necessary that every idea in the economy has the potential to be combined with every other idea, so long as the number of ideas that can be combined grows at least as quickly as the square root of the total number of ideas.

Weitzman also investigates the case where the probability a combination of ideas yields a new idea changes. For example, it may be that combination gets harder over time, or that it gets easier. This can just as easily be interpreted as a change in the R&D costs of investigating a combination. For example, if the probability two ideas yields a new idea falls from 1% to 0.1%, then you must process 10 times as many ideas to get the same yield and this costs 10 times as much.

When the probability of success varies, Weitzman shows that long-run growth is driven by two factors: (1) the long run "cost" of R&D for a single idea and (2) the flexibility of the economy to substitute capital for ideas and vice-versa. There are four cases:

	Long run cost of R&D falls to zero	Long run cost of R&D rises without limit
Capital and ideas can easily be substituted one for the other	The growth rate increases without bound	Long run steady exponential growth
Capital and ideas are not easily substitutable	Long run steady exponential growth	The growth rate falls to zero

These final results continue the theme that economic growth can be constrained by factors other the R&D process. If R&D is cheap and easy but there's just no substitute for capital (lower left box), then the capital accumulation constraint binds and steady exponential growth returns. Alternatively, if capital and ideas are largely interchangeable, then you can increasingly substitute capital investment for R&D as it gets more expensive, sustaining steady exponential growth (upper right box).

Discussion

The History of Growth

In Weitzman's model, economic growth starts slow, because the number of ideas available to combine is small relative to the size of the economy. Over time, the stock of ideas grows, and so does the growth rate, until we eventually hit a point where the number of potential combinations of ideas exceeds the economy's R&D resources. From that point onwards, the economy stabilizes at steady exponential growth. The number of potential combinations to investigate might as well be unlimited from this point forward, since the economy will never have enough resources to investigate them all.

This evolution of economic growth actually matches the actual history of economic growth relatively well. For much of human history, growth was extremely slow. We may think of this as a period when the number of ideas worth combining was small, and so every possible combination was scrutinized. Around the 1700s, economic growth began to accelerate during the industrial revolution but eventually stabilized at current levels, which have more-or-less persisted ever since. This may correspond to a transition period when the number of ideas available for combination began to rapidly increase. Indeed, with the advent of the steam engine and advances in material and chemical science, it certainly seems as if the number of technologies available for combination had notably increased. Eventually, we reached our current stage where the number of possible ideas available for combination exceeds the economy's R&D resources.

Growth Today

Another consequence of recombinant growth is that the progress of technology becomes less and less determinant over time. In the early periods of a combinatorial process, every possible combination of ideas is investigated, and so technological progress is relatively determined. Put another way, if we re-ran history from different starting conditions, the progress of technology would end up about the same.

However, once the set of ideas is large enough not all ideas are processed. Which combinations are chosen to investigate is subject to idiosyncratic and random factors. At this point, if we re-ran history starting from different starting conditions, the progress of technology could be entirely different. As Weitzman puts it "we end up on just one path taken from an almost incomprehensibly vast universe of ever-branching possibilities."

Growth in the Future

So long as we have enough ideas to get the combinatorial process started, in the long-run we will never run out of ideas to investigate. If this is a good description of our world then it appears we are fortunate to live in the case where there were enough ideas to get the combinatorial process rolling. We have experienced an acceleration of innovation converging to a steady state, not a slowdown of innovation that grinds down to zero. If this model is right, it implies we will never run out of ideas. We will only ever be constrained by how many resources we are willing to devote to R&D.

This does not necessarily mean current rates of growth will be sustained forever. It may be that combining ideas gets harder and harder over time, which implies growth can slow or even stop under some conditions, even though we never run out of ideas. This might happen, for example, if ideas grow increasingly complex when they are built from combinations of combinations of combinations … of combinations of ideas. Alternatively, it may be that combining ideas gets easier and easier over time, which implies growth can accelerate, possibly without bound. This may occur if some of our ideas (for example, artificial intelligence) make us more efficient at R&D itself.