Econ 101 - A Math Survival Guide

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1 Introduction

The introductory study of economics relies on two major ways of thinking about the world: qualitative logic and mathematical reasoning. Now just a moment, all you non-math people, don't panic just yet! For those among us who find life without a calculator unimaginable, this primer covers in brief the basics that you'll need for survival in Economics 101. In the sections below, we'll cover the math that you should be familiar with for this course. This includes some basic math (for times when you can't use a calculator), working with percentages, basic algebra and graphing.

2 Computation: Tips and Tricks

2.1 Order of Operations

The acronym GEMDAS is a variant of the memory tricks often employed by high-school math teachers for recalling the order in which to carry out math operations. Fully expanded, GEMDAS stands for Grouping, Exponents, Multiplication, Division, Addition and Subtraction–these are the standard mathematical operations listed in order of use when solving a problem. We'll make use of these extensively.

Example: *Simplification.* Reduce the following expression to its simplest terms:

$$3 - \left(x - \frac{(2+8)^2}{5}\right)$$

Answer: Starting with Grouping, we'll take care of all the operations that have to go on *inside* the parenthesis first. Within the parenthetical, we first add $2+8$, and can re-write our expression as

$$\begin{aligned} &3 - \left(x - \frac{(10)^2}{5}\right) \\ &= 3 - \left(x - \frac{100}{5}\right) \end{aligned}$$

So, we've taken care of the innermost grouping and the only exponent inside the outer parenthetical. We don't have any multiplying to do, so we'll move on to division. Dividing 100 by 5 is difficult to do right away, so we'll proceed in steps. The number 100 is the result of multiplying 10 by itself, meaning we can write $100 = 10 \times 10$. Using this fact, we can write our expression as

$$\begin{aligned}
&3 - \left(x - \frac{100}{5}\right) \\
&= 3 - \left(x - \frac{10 \times 10}{5}\right) \\
&= 3 - \left(x - \frac{10}{5} \times 10\right) \\
&= 3 - (x - 2 \times 10) \\
&= 3 - (x - 20)
\end{aligned}$$

Notice that, to do the steps above, we used the fact that multiplication happens before addition or subtraction. Finally, subtracting $x - 20$ is the same as subtracting x and *adding*[1] 20, leaving us with the final simplified form of our expression: $23 - x$.

2.2 Factoring, Multiplication and Fractions

As we did with the number 100, above, it is often the case that large numbers can be broken down into their constituent parts. We can do this in terms of both multiplication and addition.

Examples: *Multiplication.*

1. $22 = 2 \times 11$
2. $21 = 3 \times 7$
3. $108 = 4 \times 27$
4. $125 = 5 \times 25$
5. $256 = 2^8$

Examples: *Addition.*

1. $37 = 36 + 1$
2. $3 = 2 + 1$
3. $427 = 420 + 7$

[1] To see this, note that, by the order of operations, the term $(x - 20)$ is equivalent to the statement, "a number that is 20 less than whatever value x has." So, subtracting x alone would be subtracting too much–we'll eventually have to add 20 back to the result, since we're interested in subtracting a number 20 units smaller than x itself.

4. $63,545 = 63,000 + 500 + 45$
5. $2.7 = 2 + \frac{7}{10}$

Note that we should try to break up numbers into the pieces that behave nicely–i.e. pieces that we can factor easily–and the pieces that are more difficult to deal with.

Example: *Factoring and simplifying a fraction.* Let's try simplifying the fraction $\frac{427}{7}$. Rewriting 427 as $420 + 7$, we can write our fraction as

$$\frac{420 + 7}{7} = \frac{420}{7} + \frac{7}{7}$$

Simplifying further, notice that $420 = 42 \times 10$. With this in mind, now we have

$$\begin{aligned}\frac{427}{7} &= \frac{42 \times 10}{7} + \frac{7}{7} \\ &= \frac{42}{7} \times 10 + 1 \\ &= 6 \times 10 + 1 \\ &= 60 + 1 \\ &= 61\end{aligned}$$

Here, we have had to use both the addition *and* multiplication tricks to arrive at our answer. This is a helpful method to use if ever you find yourself without a calculator.

Exercises

1. Break up the following numbers into factors (numbers that multiply out to the number you see):
   (a) 22
   (b) 300
   (c) 427
2. Find at least two ways to break up each of the following numbers into additive parts (for example, $29 = 20 + 9$ or $29 = 17 + 12$):
   (a) 66
   (b) 327
   (c) 267
   (d) 4
3. Simplify the following fractions:
   (a) $\frac{427}{14}$
   (b) $\frac{21}{2}$
   (c) $\frac{37}{3}$
   (d) $\frac{399}{3}$

3 Percents and Percent Changes

In the study of economics, we're often interested in percent changes. Why? Because using percents can give us insight into whether a change is relatively large or small.

3.1 Percentages

Think, for example, about getting a raise in your paycheck. How does the raise compare to your current pay? If you're getting a raise of x dollars, we can get that number as a percent of your salary by dividing the raise by your salary. That is, take $\frac{x}{salary}$ to get the *proportion* and then multiply by 100 to get the percentage.

Example: *A raise in your salary.* Say your boss gives you a raise of \$1,000 per year. Is that a lot if you're already making \$10,000 per year? How about if you're already making \$100,000 per year?

Answer: If you're making \$10,000 per year, we can divide $\frac{1{,}000}{10{,}000}$ to obtain $\frac{1}{10}$, which is 0.1. Multiplying by 100 gives us $\frac{1}{10} \times 100 = 10\%$. If, on the other hand, you're already making \$100,000 per year, we can evaluate $\frac{1{,}000}{100{,}000} \times 100 = 1\%$. So, that \$1,000 per year raise in pay is a reasonably large chuck of your existing salary if you're making \$10,000 per year, but only a drop in the bucket if you're making \$100,000 per year.

3.2 Percent Changes

A *percent change* is a change in something measured in percentage terms. So, we need two components: the change, and the percentage. To use salaries as an example, consider a worker who makes \$45,000 per year. Then the economic downturn hits, and our worker's salary is reduced to \$42,000 per year. We can find the percent-change in the worker's salary by evaluating $\frac{42{,}000-45{,}000}{45{,}000}$, that is, we're comparing the *difference* in salary to the worker's initial salary. This leaves us with $\frac{-3{,}000}{45{,}000}$, since our worker experienced a decrease in salary (it'd be positive for an increase in pay).

Using the math we reviewed earlier, we can simplify: $\frac{-3{,}000}{45{,}000} = \frac{-3}{45} = \frac{-3}{9\times5} = \frac{-1}{3\times5} = \frac{-1}{15}$. Multiplying by 100 gives us the percent change:

$$\begin{aligned}
-\frac{1}{15} \times 100 &= -\frac{1}{3} \times \frac{100}{5} \\
&= -\frac{1}{3} \times 20 \\
&= -\frac{18+2}{3} \\
&= -\frac{18}{3} - \frac{2}{3} \\
&= -(6 + \frac{2}{3}) \\
&= -6.\bar{6}\,\%
\end{aligned}$$

That is, our worker has experienced a $6.\bar{_6}\%$ *drop* in pay.

4 Algebra

The algebra encountered in basic economics coursework involves solving equations for variables with names like "price" and "quantity." Let's work through a few examples.

4.1 One Equation and One Unknown Value

Let's say a consumer buys distilled water according to the rule $Q = 30 - \frac{1}{2}P$, where Q is the quantity of water purchased in fluid ounces and P is the price per ounce of water, given in *cents*. If we observe this consumer purchasing precisely 22 oz of distilled water, what price is she facing?

We can set up the problem as:

$$22\text{oz} = 30 - \frac{1}{2}\, P \text{ per oz.}$$

Now we can add $\frac{1}{2}\, P$ to both sides of the equation and subtract 22 from both sides of the equation. This gives us:

$$\frac{1}{2}P = 30{-}22 = 8$$
$$\Leftrightarrow$$
$$P = 2 \times 8 = 16 \text{ cents per oz.}$$

where we have multiplied both sides of the equation by 2 to get our result.

4.2 Two Equations and Two Unknown Values

More commonly, we will see *systems* of equations–that is, equations that tell the story of more than one variable. Mostly, we will be interested in the following sort of question:

Can we find values of P and Q for which the statements $Q = 31 - \frac{1}{2}P$ and $P = 4Q + 2$ are both true?

Let's find out if we can. Start by noticing that $P = 4Q + 2$ can replace the value P in $Q = 31 - \frac{1}{2}P$. Doing the substitution gives us

$$Q = 31 - \frac{1}{2} \times (4Q + 2)$$

Using the order of operations, we can distribute the fraction $-\frac{1}{2}$ to obtain

$$\begin{aligned} Q &= 31 - \frac{1}{2} \times 4Q - \frac{1}{2} \times 2 \\ &= 31 - 2Q - 1 \\ &= 30 - 2Q \end{aligned}$$

Now we can add $2Q$ to both sides, and we can re-write the whole thing as

$$3Q = 30$$

Dividing both sides by 3 tells us that $Q = 10$. So, we've found that $Q = 10$, but what about P? If $Q = 10$, then what must P be in order to make both equations true? Using $Q = 10$, we can write out

$$P = 4 \times 10 + 2 = 42$$

So, in conclusion, we can say that both equations are true when $Q = 10$ and $P = 42$.

4.3 Exercises

1. Solve the following equations for the unknown variable. If there is no way to find all the unknown values, see if you can figure out why this is the case.

   (a) $21x + 20 = 23$

   (b) $\frac{4}{x} = 2$

   (c) $x^2 + 7 = 71$

   (d) $(x + 2y)^3 = 8$

2. Solve the following systems of equations for all variables:

   (a) Two-equation system:

$$x + y = 2 \quad (1)$$
$$\frac{1}{2}x - y = 0 \quad (2)$$

   (b) Two-equation system:

$$x + 4 = 14y \quad (3)$$
$$\frac{x}{y} = 7 \quad (4)$$

   (c) Three-equation system:

$$x + y + z = 0 \quad (5)$$
$$x - Z = 1 \quad (6)$$
$$y + 5 = 1 \quad (7)$$

5 Graphs and their (dis)contents

5.1 Graphs as Visual Aids

Consider the equation $y = 2x + 3$. How can we describe it?

We might use a chart:

x	y
-5	-7
-3	-3
-1	1
0	3
1	5
3	9
5	13

Table 1: Values of x and y satisfying $y = 2x+3$

Or we could use a graph:

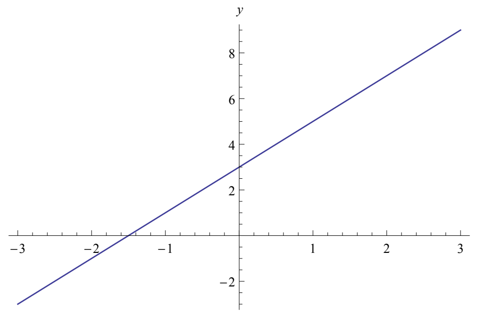

Figure 1: A graph of $y = 2x + 3$

We can see that this is a handy way to visualize the equation–it allows us to quickly see the pattern of values for y across a range of values for x.

5.2 Parts of a Graph

a Axis – the plural is "axes," and the term describes the measurement of the value of any variable you care about. We'll think of graphs as having one axis per variable.

b Origin – this is where all the axes meet.

c Lines – representations of the relationship between variables measured on different axes.

d Coordinates – instructions for the placement (or location) of a point on the graph, measured by distance away from the origin along all axes.

5.3 A Note on Coordinates

We can label points on our graphs by utilizing a system of coordinates to describe their location relative to the origin. If we call the vertical axis y and the horizontal axis x, the standard way to write coordinates is (x, y). So, in Table 1, for example, the point (1, 5) on the line $y = 2x + 3$ is located one unit away from the origin along the x-axis and five units away from the origin along the y-axis. Intuitively, since the origin–where all the axes meet–is the main point of reference for the coordinate system, we can label the origin itself as the point (0, 0). Legend has it that René Descartes (1596 - 1650), the French philosopher, mathematician, writer and alleged recreational duelist whose name is used to describe the standard two-axis graph[2] (called the "Cartesian graph") invented the standard coordinate

[2] He's also the guy who made the statement, "*Cognito, ergo sum*" ("I think, therefore I am").

system while in the hospital recovering from wounds sustained in a swordfight.[3] Staring at the tiled ceiling above his hospital bed, he noticed a fly perched there. To describe the location of the fly, Descartes realized, he could pinpoint it by comparing its position to the corner of the tile on which it had landed. For example, he might have thought of the fly as being located two centimeters to the right of the corner of the tile and four centimeters up from the corner. With those two measurements, one for each dimension of the space over which the fly was crawling, Descartes could fully describe its location.

5.4 Graphing Lines

When graphing, we can think of a *line* as having the form $y = mx + b$. As usual, the x is one variable, and the modifications rendered by m and b yield the result to which we'll refer as y. That is, $mx + b$ is really a set of instructions for getting from x to y. The lines in which we're interested have two parts:

1. Slope – the rate and direction with which y changes when x changes in a given direction.
2. Intercept – a description of where the line hits one of the axes.

Let's look more closely at the equation for a line:

$$y = \underbrace{m}_{\text{slope}} x + \underbrace{b}_{\text{y-intercept}}$$

We can define these concepts each in turn:

1. We can recall that slope is often described as "rise over run" on a standard graph. More formally, we can think of this as $\frac{\textit{change in } y}{\textit{change in } x}$, where y is measured on the vertical axis (up-and-down) and x is measured on the horizontal axis (side-to-side). That is, the concept of slope describes the "moving" part of the relationship between x and y.

   Example: *Average speed of a car.* Imagine that you travel 216 miles by car in 3 hours. What is your (average) speed in miles per hour?

   Answer: To get the rate of change of position (distance from your starting point) per hour (so, per unit that time changes), we can set up the fraction $\frac{216\, miles}{3\, hours}$, which reduces to $\frac{72\, miles}{hour}$ or 72 mph.

2. The intercept of a line–that is, the point at which it crosses an axis–happens when the variable corresponding to the axis *not* being crossed is zero. In coordinate terms, for example, the $y - intercept$, where the graphed line crosses the y-axis, is located at (0, 3)–that is, when $x = 0$ and $y = 3$. To see this on the graph in Figure 1, track along the x axis as the line approaches the intercept point (0, 3). Measured along the x axis, we arrive at the position of the y-intercept when x is precisely zero.

   Example: *Algebra on the graph.* Now that you know the y-intercept of the graph $y = 2x + 3$, find the $x - intercept$.

[3] It is unknown, however, whether Descartes was victorious in this enterprise.

Answer: as mentioned above, set the off-axis value to zero–when $y = 0$, we get

$$\begin{gathered} 0 = 2x + 3 \\ \Leftrightarrow \\ 2x = -3 \\ \Leftrightarrow \\ x = \frac{-3}{2} \end{gathered}$$

After the dust settles from our algebra, we arrive at the result that the line crosses the x-axis when $y = 0$ and $x = \frac{-3}{2}$

5.5 Some Thoughts About Slopes

As above, we can generally think of a *slope* as a rate of change. It could describe the rate of change in elevation while hiking up the side of a mountain; it could describe the change in location over time as the speed of travel; it could even represent the change in speed over time–acceleration.[4] As a matter of fact, if we look hard enough, pretty soon we'll see just about every rate as a slope. Dollars-per-minute phone call charges, dollars-per-megabit data downloads on your smartphone and 20-cents-per-message texting can all be thought of as slopes. As budding economists, we can think of these three examples as the rate of change in your smartphone bill per additional unit of call time, download time and text message sent.

5.6 Calculating Areas on a Graph

The typical graphs that we will encounter in economics will generally look like standard shapes found in any geometry textbook. As a result, if we wanted to calculate the areas represented by the shapes on our graphs, we can use the basic tools of geometry to do so. Recall two basic definitions:

Definition: *Area of a Triangle*

$$\text{Area of a Trianlge} = \frac{1}{2} \times base \times height$$

Definition: *Area of a Rectangle*

$$\text{Area of a Rectangle} = base \times height$$

Let's put them to use. In Figure 2, below, note that the blue line creates a right triangle where it crosses the y and x axes.

Exercise: *Area of a Triangle.* Find the area of the right triangle formed by the y-axis, the x-axis and the line with equation $y = -\frac{1}{2}x + 3$.

[4] As a point of information, it turns out that physicists refer to the rate of change of acceleration of an object as its "jerk." Go figure.

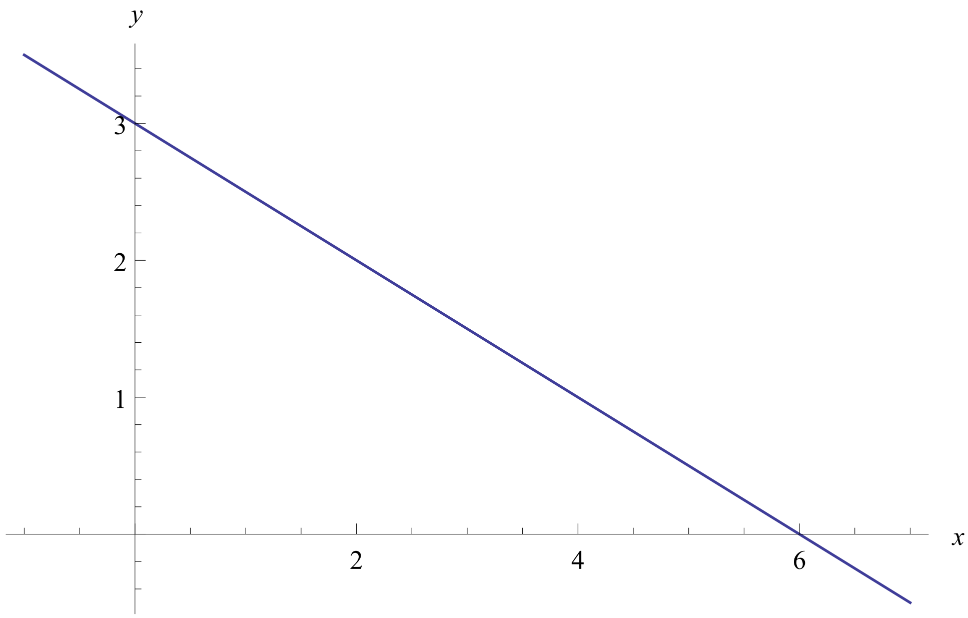

Figure 2: Plot of $y = -\frac{1}{2}x + 3$

Answer: Using the definition that the area of a triangle is one-half of the product of the base and height, we can note that the *base* of the triangle is given by the segment of the x-axis from the origin ($x = 0$) to $x = 6$, or between (0, 0) and (6, 0). Similarly, the height of the triangle is given by the segment of the y-axis from the origin to $y = 3$, or from (0, 0) to (0, 3). So, $base \times height = 6 \times 3 = 18$, thus $\frac{18}{2} = 9$, which is the area of our triangle.

5.7 Exercises

1. For the equation $y = -\frac{1}{3}x + 30$, create a table like Table 1 for values of x -3, -1, 0, 1, 3, 6, 30, and 90, and then graph the result as a solid line on a standard graph.

2. For the equation $y = 3x$, carry out the same steps as in the problem above.

3. For the two equations above, find the coordinates of the point at which they intersect. Hint: this will be given by the values of x and y for which *both* equations are true.

4. What is the slope of the graph in Problem 1?

5. What is the y-intercept of the graph in Problem 2?

6. What is the area of the shape formed by the x axis and the two equations? That is, find the area that is above the x-axis, below the graph $y = 3X$ and below the graph of $y = -\frac{1}{3}x + 30$.

7. What is the area of the square whose corners are the origin, (0, 0), and the point of intersection of the graphs of the two equations?