1 Deriving The Formula For Savings With Annual Contributions

Say you contribute c to a savings account with interest rate $r = R/100$ each year, for Y years.

The first c you contribute accumulates interest for Y years, and is worth $c \cdot (1+r)^Y$ after Y years of saving.

The second c you contribute has one less year to accumulate interest, so it is worth $c \cdot (1+r)^{Y-1}$ after Y years of saving.

The third c you contribute has two less years to accumulate interest, so it is worth $c \cdot (1+r)^{Y-2}$ after Y years of savings.

This pattern continues all the way down to the c you contribute in the year before retirement. It has one year to accumulate interest, so it is worth $c \cdot (1+r)$ when you retire.

Finally, your last contribution, which I assume occurs on the day you retire, has no time to accumulate interest, so it is just worth c.

Taking all this together, the balance B of your contributions at the time of retirement is:

$$c \cdot (1+r)^Y + c \cdot (1+r)^{Y-1} + c \cdot (1+r)^{Y-2} + \cdots + c \cdot (1+r) + c = B$$

Now we perform some clever algebraic manipulations. First, multiply each side by $(1 - (1+r))$:

$$(1-(1+r))\left[c \cdot (1+r)^Y + c \cdot (1+r)^{Y-1} + c \cdot (1+r)^{Y-2} + \cdots + c \cdot (1+r) + c\right] = B\,(1-(1+r))$$

Next, expand the multiplication on the left-hand side. The terms multiplied by 1 are unchanged, while the terms multiplied by $-(1+r)$ switch to negative and have an additional $(1+r)$ multiplying them:

$$\left\{ \begin{array}{c} c \cdot (1+r)^Y + c \cdot (1+r)^{Y-1} + c \cdot (1+r)^{Y-2} + \cdots + c \cdot (1+r) + c \\ -c \cdot (1+r)^{Y+1} - c \cdot (1+r)^{Y} - c \cdot (1+r)^{Y-1} - \cdots - c \cdot (1+r)^2 - c \cdot (1+r) \end{array} \right\} = B\,(1-(1+r))$$

Notice that the term $c \cdot (1+r)^Y$ has a partner $-c \cdot (1+r)^Y$, the term $c \cdot (1+r)^{Y-1}$ has a partner $-c \cdot (1+r)^{Y-1}$. In fact, every term has a cancelling partner except for the c and the $-c \cdot (1+r)^{Y+1}$ terms. If we cancel all these terms we are left with:

$$c - c \cdot (1+r)^{Y+1} = B\,(1-(1+r))$$

Meanwhile, on the right hand side, notice that $1-(1+r) = -r$. This means we can rewrite the equation as:

$$c - c \cdot (1+r)^{Y+1} = -rB$$

Factor out the c terms and multiply each side by -1 to obtain

$$c\left[(1+r)^{Y+1} - 1\right] = rB$$

Finally, divide each side by r and obtain the result:

$$\frac{c}{r}\left[(1+r)^{Y+1}-1\right]=B$$